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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF CERTAIN CLASSES OF DYNAMIC PROBLEMS OF NONLINEAR ELASTICITY THEORY

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There is proved an existence and uniqueness theorem for the solutions of dynamic problems of the nonlinear elasticity theory of finite deformations in the Sobolev space $H_1(\Omega)$. The solution of an analogous problem for classical elasticity theory with small deformations and a linear elasticity law is based on the Korn inequality [1]. Questions of the existence and uniqueness of solutions of linear and quasilinear equations of evolutionary type have been studied in [2 – 4].

1. Formulation of the problem. Let a body in the natural undeformed state occupy a domain Ω in R^3 with the boundary $\Gamma = \partial\Omega$. A one-parameter group of mappings gt: $\Omega \to R^3$ gives the motion of an elastic body

$$g^t: \mathbf{a} \to \mathbf{r} = \mathbf{a} + \mathbf{u}(\mathbf{a}, t), \quad \mathbf{r} \in R^3 \tag{1.1}$$

Here $\mathbf{a}\,(a_1, a_2, a_3) \in \Omega$ are Lagrange coordinates of points of the body in a certain inertial coordinate system.

Assuming the body homogeneous and isotropic, $\mathbf{u} \in V(\Omega)$, $\mathbf{u}^\cdot \in H_0(\Omega)$, where $V(\Omega)$ and $H_0(\Omega)$ are the Banach and Hilbert spaces, respectively, we represent the functionals of the kinetic and potential elastic energies as

$$T[\mathbf{u}^\cdot] = \frac{1}{2}\rho(\mathbf{u}^\cdot, \mathbf{u}^\cdot) = \frac{1}{2}\rho\int_\Omega \mathbf{u}^{\cdot 2}\, da \quad (\mathbf{u}^\cdot = D_t\mathbf{u}) \tag{1.2}$$

$$\mathrm{E}[\mathbf{u}] = \int_\Omega e(\mathrm{I}_E, \mathrm{II}_E, \mathrm{III}_E)\, da \quad (da = da_1 da_2 da_3)$$

Here ρ is the density of the body in the natural state, which we shall henceforth consider to be one, e: $R^3 \to R_t^1$ is the specific elastic strain energy, I_E, II_E, III_E are invariants of the finite strain tensor [5]

$$E = {}^1/_2(U + U^T + U^T U), \quad U = (u_{ij} = \partial u_i / \partial a_j) \tag{1.3}$$

The functional space $V = \{\mathbf{u}: \mathrm{E}[\mathbf{u}] < +\infty\}$ is the Sobolev space $(W_p^1(\Omega))^3$ or a manifold therein, where [6]

$$W_p^1(\Omega) = \Big\{u: \Big(\sum_{\alpha \leqslant 1}\int_\Omega |D_a^\alpha u|^p\, da\Big)^{1/p} < \infty\Big\}$$

Let us assume that $V \subset (W_2^1(\Omega))^3 \equiv H_1(\Omega)$ and that there is an element in $(W_{2-\alpha}^1(\Omega))^3$ $(0 < \alpha \leqslant 1)$ on which the potential energy $\mathrm{E}[\mathbf{u}]$ is not defined. In this sense the space $H_1(\Omega)$ is the broadest space in which the potential energy functional is defined.

The equations of elastic body motion follow from the D'Alembert — Lagrange principle of possible displacements in the form

$$(\mathbf{u}^{\cdot\cdot}, \delta\mathbf{u}) + (\nabla \mathrm{E}\,[\mathbf{u}], \delta\mathbf{u}) - (\mathbf{f}, \delta\mathbf{u}) - (\mathbf{F}, \delta\mathbf{u})_\Gamma = 0, \quad \delta\mathbf{u} \in V \tag{1.4}$$

$$(\nabla \mathrm{E}\,[\mathbf{u}], \delta\mathbf{u}) = \int\limits_\Omega \sum_{i,j=1}^{3} \frac{\partial e}{\partial u_{ij}} \delta u_{ij}\, da, \quad (\mathbf{u}^{\cdot\cdot}, \delta\mathbf{u}) = \int\limits_\Omega \mathbf{u}^{\cdot\cdot}\delta\mathbf{u}\, da$$

$$(\mathbf{f}, \delta\mathbf{u}) = \int\limits_\Omega \mathbf{f}\delta\mathbf{u}\, da, \quad (\mathbf{F}, \delta\mathbf{u})_\Gamma = \int\limits_\Gamma \mathbf{F}\delta\mathbf{u}\, d\sigma$$

Here $\mathbf{F}$ is the external surface force, $\mathbf{f}$ is the internal force, $\delta\mathbf{u}$ is the vector of possible displacements. It is assumed that the external surface forces are given on a part of the boundary Γ_F, while displacements

$$\mathbf{a} \in \Gamma_U, \mathbf{u} = \mathbf{U}\,(\mathbf{a}, t) \quad (\Gamma_U \cap \Gamma_F = \phi, \Gamma_U \cup \Gamma_F = \Gamma) \tag{1.5}$$

are given on the rest.

The linear manifold $V_0 = \{\mathbf{u}\colon \mathbf{u} \in V,\ \mathbf{u}\,|_{\Gamma_U} = \mathbf{U}\,(\mathbf{a},\ t)\} \subset V$ will be a configuration space of the system in this case.

The function $\mathbf{U}\,(\mathbf{a},\ t)$ should satisfy conditions of the theorem on traces on Γ_U : if Ω is an open domain in R^3 whose boundary Γ is an infinitely differentiable oriented manifold of dimensionality 2 relative to which Ω is found locally on one side, then the trace $\gamma\mathbf{u}$ of the function $\mathbf{u} \in H_1\,(\Omega)$ is a function belonging to $H_{1/2}\,(\Gamma)$ and the mapping $H_1\,(\Omega) \to H_{1/2}\,(\Gamma)$ is linear and continuous, i. e.

$$\|\,\gamma\mathbf{u}\,\|_{H_{1/2}(\Gamma)} \leqslant c\,\|\,\mathbf{u}\,\|_{H_1(\Omega)} \tag{1.6}$$

Here the constant c is independent of $\mathbf{u}$ [6]. Furthermore, we assume that there exists a function $\mathbf{u}_0\,(\mathbf{a},\ t) \in V$ and $\gamma\mathbf{u}_0\,|_{\Gamma_U} = \mathbf{U}\,(\mathbf{a},\ t)$. The time is considered as a parameter here. The mentioned conditions for smoothness of the function $\mathbf{U}\,(\mathbf{a},\ t)$ are usually satisfied in applications.

Let us set $\mathbf{v} = \mathbf{u} - \mathbf{u}_0$ and let us define the linear space

$$V_0 = \{\mathbf{v} : \mathbf{v} \in V, \mathbf{v}\,|_{\Gamma_U} = 0\} \subset H_1\,(\Omega)$$

The possible displacements of the system are $\delta\mathbf{u} = \delta\mathbf{v} \in V_0$. The space V_0 is a configuration space of the system, and the direct product $V_0 \times H_0$ is a phase space of the system $(H_0 = (L_2\,(\Omega))^3)$.

The problem (1.4) becomes

$$(\mathbf{v}^{\cdot\cdot}, \delta\mathbf{v}) + (\nabla \mathrm{E}\,[\mathbf{v} + \mathbf{u}_0], \delta\mathbf{v}) - (\mathbf{f}_0, \delta\mathbf{v}) - (\mathbf{F}, \delta\mathbf{v})_\Gamma = 0, \quad \forall\delta\mathbf{v} \in V_0 \tag{1.7}$$

$$\mathbf{v}\,(\mathbf{a}, t)\,|_{\Gamma_U} = 0 \quad (\mathbf{f}_0 = \mathbf{f} - \mathbf{u}_0^{\cdot\cdot})$$

$$\mathbf{v}\,(\mathbf{a}, 0) = \mathbf{u}\,(\mathbf{a}, 0) - \mathbf{u}_0\,(\mathbf{a}, 0) \equiv \mathbf{v}_0\,(\mathbf{a}) \in V_0$$

$$\mathbf{v}^{\cdot}\,(\mathbf{a}, 0) = \mathbf{u}^{\cdot}\,(\mathbf{a}, 0) - \mathbf{u}_0^{\cdot}\,(\mathbf{a}, 0) \equiv \mathbf{v}_0^{\cdot}\,(\mathbf{a}) \in H_0$$

The existence and uniqueness of solutions of problem (1.7) are examined below. All the integrals in (1.7) will have meaning if

$$\mathbf{v}^{\cdot\cdot},\ \nabla \mathrm{E}\,[\mathbf{v} + \mathbf{u}_0],\quad \mathbf{f}_0 \in V',\quad \mathbf{F} \in H_{-1/2}(\Gamma) \tag{1.8}$$

Here V' and $H_{-1/2}(\Gamma)$ are spaces conjugate to the spaces V_0 and $H_{1/2}\,(\Gamma)$ respectively [6].

2. **Existence theorem of the solutions.** Let the mapping $\nabla E[\mathbf{u}]: \ V_0 \to V'$ satisfy the Lipschitz condition

$$\| \nabla E[\mathbf{u}'] - \nabla E[\mathbf{u}''] \|_{-1} \leqslant L \| \mathbf{u}' - \mathbf{u}'' \|_1, \quad \forall \mathbf{u}', \mathbf{u}'' \in V_0, \quad L > 0 \tag{2.1}$$

Here $\|\cdot\|_{-1}$, $\|\cdot\|_1$ are norms in $H_{-1}(\Omega)$ and $H_1(\Omega)$, respectively. We assume that for any $\varkappa > 0$ there are such positive constants k_1 and k_2 that

$$k_1 \| \mathbf{u} \|_1^2 \leqslant E[\mathbf{u}] + \varkappa \| \mathbf{u} \|_0^2 \leqslant k_2 \| \mathbf{u} \|_1^2, \quad \forall \mathbf{u} \in V_0 \tag{2.2}$$

Here $\|\cdot\|_0$ is the norm in $H_0(\Omega)$.

With respect to the external forces and displacements on part of the boundary, we assume compliance with the conditions

$$\begin{aligned} & f_i, f_i^{\cdot} \in L_2(Q), \quad Q = \Omega \times [0, T] \quad (T > 0) \\ & F_i, F_i^{\cdot} \in L_2(\Sigma), \ \Sigma = \Gamma \times [0, T] \qquad (i = 1, 2, 3) \\ & U_i^*, U_i^{*\cdot}, U_i^{*\cdot\cdot}, U_i^{*\cdot\cdot\cdot} \in L_2(0, T; H_{1/2}(\Gamma)) \end{aligned} \tag{2.3}$$

Here U_i^* agrees with U_i on Γ_U.

Theorem. If the potential energy functional satisfies conditions (2.1) and (2.2), and the external forces and displacements $\mathbf{U}$ on Γ_U satisfy the conditions (2.3) and

$$\mathbf{u}(\mathbf{a}, 0) \in V_0(\Omega), \quad \mathbf{u}^{\cdot}(\mathbf{a}, 0) \in H_0(\Omega) \tag{2.4}$$

then the problem (1.7) has a solution and

$$\begin{aligned} & \mathbf{v}(\mathbf{a}, t) \in L_\infty(0, T; V_0), \quad \mathbf{v}^{\cdot}(\mathbf{a}, t) \in L_\infty(0, T; H_0) \\ & \mathbf{v}^{\cdot\cdot}(\mathbf{a}, t) \in L_\infty(0, T; V') \end{aligned} \tag{2.5}$$

Note 1. The first two conditions in (2.3) can be weakened by assuming

$$\mathbf{f}, \mathbf{f}^{\cdot} \in L_2(0, T; H_{-1}(\Omega)), \quad \mathbf{F}, \mathbf{F}^{\cdot} \in L_2(0, T; H_{-1/2}(\Gamma))$$

However, the first two conditions of (2.3) are completely adequate for physical problems.

Note 2. The function U^* in the last condition of (2.3) is a continuation of the function U given on Γ_U to the whole manifold Γ. It follows from the last condition in (2.3) and the theorem about traces that there exists a function $\mathbf{u}_0(\mathbf{a}, t)$ such that

$$\mathbf{u}_0, \mathbf{u}_0^{\cdot}, \mathbf{u}_0^{\cdot\cdot}, \mathbf{u}_0^{\cdot\cdot\cdot} \in L_2(0, T; H_1(\Omega))$$

Then the imbeddings for the initial conditions in (1.7) are valid, and $\mathbf{f}_0$ from (1.7) satisfies the condition $f_{0i}, f_{0i}^{\cdot} \in L_2(Q)$ $(i = 1, 2, 3)$.

Note 3. In the case of a linear elasticity law and small deformations, the inequality (2.2) follows from the Korn inequality [1].

The proof of the theorem is based on applying the Galerkin method to construct successive approximations of the solution in finite-dimensional spaces, to estimate their boundedness in the system phase space, and to prove that the limit function satisfies (1.7).

Construction of the approximate solutions. By using the property of separability of the space V_0, we select the basis $\{\mathbf{w}_m\}_{m=1}^{\infty}$ orthonormalized in the sense of $H_0(\Omega)$ by assuming $\mathbf{w}_1 = \mathbf{v}_0 / \| \mathbf{v}_0 \|_0$. Let $V_1^{(m)}$

be a finite space of linear combinations of the vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$, $V_0^{(m)} \subset V_0$.

Let us call the function $\mathbf{v}^{(m)} \in V_0^{(m)}$ satisfying the equation

$$(\mathbf{v}^{\cdot\cdot(m)}, \delta\mathbf{v}) + (\nabla E[\mathbf{v}^{(m)} + \mathbf{u}_0], \delta\mathbf{v}) - (\mathbf{f}_0, \delta\mathbf{v}) - (\mathbf{F}, \delta\mathbf{v})_\Gamma = 0, \quad \forall \delta\mathbf{v} \in V_0^{(m)} \tag{2.6}$$

and the initial conditions $\mathbf{v}^{(m)}(\mathbf{a}, 0) = \mathbf{v}_0(\mathbf{a})$, $\mathbf{v}^{\cdot}(\mathbf{a}, 0) = \mathbf{v}_{0m}^{\cdot}(\mathbf{a})$ and approximate solution of (1.7) ($\mathbf{v}_{0m}^{\cdot}(\mathbf{a})$ is the projection of $\mathbf{v}_0^{\cdot}$ on $V_0^{(m)}$). Equation (2.6) is equivalent to a system of ordinary differential equtions of order $2m$. Setting

$$\mathbf{v}^{(m)} = \sum_{i=1}^{m} q_{im}(t)\, \mathbf{w}_i$$

and replacing $\delta\mathbf{v}$ in (2.6) by w_i $(i = 1, \ldots, m)$, we arrive at the system

$$\ddot{q}_{im} + \left(\nabla E\left[\sum_{k=1}^{m} q_{km}\mathbf{w}_k + \mathbf{u}_0\right], \mathbf{w}_i\right) - (\mathbf{f}_0, \mathbf{w}_i) - (\mathbf{F}, \mathbf{w}_i)_\Gamma = 0 \quad (i = 1, \ldots, m) \tag{2.7}$$

We show that the system (2.7) satisfies the Lipschitz condition, which means that there is a unique solution in a certain segment $[0, T_m]$. Using the Euclidean metric in the space $V_0^{(m)}$ and (2.1), we arrive at the estimate

$$\begin{aligned}\left[\sum_{i=1}^{m}\left(\nabla E\left[\sum_{k=1}^{m} q'_{km}\mathbf{w}_k + \mathbf{u}_0\right] - \nabla E\left[\sum_{k=1}^{m} q''_{km}\mathbf{w}_k + \mathbf{u}_0\right], \mathbf{w}_i\right)^2\right]^{1/2} \leqslant \\ \left[\sum_{i=1}^{m}\left\|\nabla E\left[\sum_{k=1}^{m} q'_{km}\mathbf{w}_k + \mathbf{u}_0\right] - \nabla E\left[\sum_{k=1}^{m} q''_{km}\mathbf{w}_k + \mathbf{u}_0\right]\right\|_{-1}^2 \times \\ \|\mathbf{w}_i\|_1^2\right]^{1/2} \leqslant L \max_{1\leqslant i\leqslant m}\|\mathbf{w}_i\|_1 \sqrt{m}\left\|\sum_{k=1}^{m}(q'_{km} - q''_{km})\mathbf{w}_k\right\|_1 \leqslant \\ L \max_{1\leqslant i\leqslant m}\|\mathbf{w}_i\|_1^2\, m\left[\sum_{k=1}^{m}(q'_{km} - q''_{km})^2\right]^{1/2}\end{aligned} \tag{2.8}$$

It hence follows that the Lipschitz condition is satisfied with a constant for the system (2.7) (let us note that (2.7) is a second order equation)

$$L(m) = \max(1, Lm \max_{1\leqslant i\leqslant m}\|\mathbf{w}_i\|_1^2).$$

The constant $L(m)$ grows as in the dimensionality m of the space increases, and also because of the growth of $\max_{1\leqslant i\leqslant m}\|\mathbf{w}_i\|_1^2$.

Therefore, according to the existence and uniqueness theorem for solutions of ordinary differential equations, the solution (2.7) exists and is unique on some segment $[0, T_m]$ and $T_m \to 0$ as $m \to \infty$.

Uniform boundedness of the solution. The possibility of continuing the solutions (2.7) in a certain segment $[0, T]$ independent of m results from the uniform boundedness of all the solutions independently of the number m and the system phase space $V_0 \times H_0$.

Let us replace $\delta\mathbf{v}$ in (2.6) by $\mathbf{v}^{\cdot(m)}$ and let us integrate the expression obtained with respect to the time between 0 and t. We obtain

$$\frac{1}{2}\|\mathbf{v}^{\cdot(m)}\|_0^2 - \frac{1}{2}\|\mathbf{v}^{\cdot(m)}(0)\|_0^2 + E[\mathbf{v}^{(m)} + \mathbf{u}_0] - E[\mathbf{v}_0 + \mathbf{u}_0(0)] = \tag{2.9}$$

$$\int_0^t [(\mathbf{f}_0, \mathbf{v}^{\cdot(m)}) + (\mathbf{F}, \mathbf{v}^{\cdot(m)})_\Gamma + (\nabla \mathrm{E}\,[\mathbf{v}^{(m)} + \mathbf{u}_0], \mathbf{u}_0^{\cdot})]\, dt$$

Using the Young inequality, integration by parts, and conditions (2.3) and (2.4), we estimate the integral in the right side of (2.9)

$$|I| \leqslant \frac{1}{2\varepsilon}(\|\mathbf{f}_0\|_{-1}^2 + \|\mathbf{F}\|_{\Gamma,-1/2}^2) + \frac{\varepsilon}{2}(1 + c^2)\|\mathbf{v}^{(m)}\|_{\mathrm{I}}^2 + \quad (2.10)$$
$$\frac{1}{2}[\|\mathbf{f}_0(0)\|_{-1}^2 + (1 + c^2)\|\mathbf{v}_0\|_{\mathrm{I}}^2 + \|\mathbf{F}(0)\|_{\Gamma,-1/2}^2] +$$
$$\int_0^t [\|\mathbf{f}_0^{\cdot}\|_{-1}^2 + \|\mathbf{F}^{\cdot}\|_{\Gamma,-1/2}^2 + (1 + c^2 + L^2)\|\mathbf{v}^{(m)}\|_{\mathrm{I}}^2 +$$
$$\|\mathbf{u}_0^{\cdot}\|_{\mathrm{I}}^2 + L^2\|\mathbf{u}_0\|_{\mathrm{I}}^2]\, dt$$

Here ε is any positive number, c is the constant in the inequality (1.6), and $\|\cdot\|_{\Gamma,-1/2}$ is the norm in $H_{-1/2}(\Gamma)$.

According to (2.2)

$$\mathrm{E}\,[\mathbf{v}^{(m)} + \mathbf{u}_0] \geqslant k_{\mathrm{I}}\|\mathbf{v}^{(m)} + \mathbf{u}_0\|_{\mathrm{I}}^2 - \varkappa\|\mathbf{v}^{(m)} + \mathbf{u}_0\|_0^2 \quad (2.11)$$

$$\|\mathbf{v}^{(m)} + \mathbf{u}_0\|_0^2 \leqslant 2\|\mathbf{v}^{(m)}(0) + \mathbf{u}_0(0)\|_0^2 + c_1\int_0^t \|\mathbf{v}^{\cdot(m)} + \mathbf{u}_0^{\cdot}\|_0^2\, dt \quad (2.12)$$

Here c_1 is a positive constant, common to all functions from $H_0(\Omega)$.

The inequalities (2.10) – (2.12) permit obtaining the following estimate from (2.9)

$$\frac{1}{2}\|\mathbf{v}^{\cdot(m)}\|_0^2 + k_{\mathrm{I}}\|\mathbf{v}^{(m)} + \mathbf{u}_0\|_{\mathrm{I}}^2 - \frac{\varepsilon}{2}(1 + c^2)\|\mathbf{v}^{(m)}\|_{\mathrm{I}}^2 \leqslant \quad (2.13)$$
$$\Psi_{\mathrm{I}}(t) + \frac{1}{2}\|\mathbf{v}^{\cdot(m)}(0)\|_0^2 + \int_0^t [(1 + c^2 + L^2)\|\mathbf{v}^{(m)}\|_1^2 +$$
$$2\varkappa c_{\mathrm{I}}\|\mathbf{v}^{\cdot(m)}\|_0^2]\, dt$$
$$\Psi_{\mathrm{I}}(t) = \mathrm{E}\,[\mathbf{v}_0 + \mathbf{u}_0(0)] + \frac{1}{2\varepsilon}(\|\mathbf{f}_0\|_{-1}^2 + \|\mathbf{F}\|_{\Gamma,-1/2}^2) +$$
$$\frac{1}{2}[\|\mathbf{f}_0(0)\|_{-1}^2 + (1 + c^2)\|\mathbf{v}_0\|_{\mathrm{I}}^2 + \|\mathbf{F}(0)\|_{\Gamma,-1/2}^2] +$$
$$2\varkappa\|\mathbf{v}_0 + \mathbf{u}_0(0)\|_0^2 + \int_0^t (\|\mathbf{f}_0^{\cdot}\|_{-1}^2 + \|\mathbf{F}^{\cdot}\|_{\Gamma,-1/2}^2 +$$
$$\|\mathbf{u}_0^{\cdot}\|_{\mathrm{I}}^2 + L^2\|\mathbf{u}_0\|_{\mathrm{I}}^2 + 2\varkappa c_1\|\mathbf{u}_0^{\cdot}\|_0^2)\, dt$$

Taking into account the inequality $\|\mathbf{v}^{(m)} + \mathbf{u}_0\|_1^2 \geqslant 1/2\,\|\mathbf{v}^{(m)}\|_1^2 - \|\mathbf{u}_0\|_1^2$, we select ε such that $1/2\,[k_1 - \varepsilon(1 + c^2)] = \mu_1 > 0$. Then we have form(2.13) and the inequality $\|\mathbf{v}^{\cdot(m)}(0)\|_0 \leqslant \|\mathbf{v}_0\|_0$

$$\alpha_{\mathrm{I}}\varphi_m(t) \leqslant \Psi_2(t) + \alpha_2\int_0^t \varphi_m(t)\, dt \quad (\varphi_m(t) = \|\mathbf{v}^{\cdot(m)}\|_0^2 + \|\mathbf{v}^{(m)}\|_{\mathrm{I}}^2) \quad (2.14)$$

$$\alpha_1 = \min(1/2, \mu_{\mathrm{I}}), \quad \alpha_2 = \max(1 + c^2 + L^2, 2\varkappa c_{\mathrm{I}})$$
$$\Psi_2(t) = \Psi_{\mathrm{I}}(t) + 1/2\|\mathbf{v}_0^{\cdot}\|_0^2 + k_1\|\mathbf{u}_0\|_1^2$$

The function Ψ_2 is positive, bounded, and independent of m. Using the Gronwall inequality for the estimate (2.14) [3], we have

$$\varphi_m(t) \leqslant \alpha_1^{-1}\Psi_2{}^* \exp(\alpha_2\alpha_1^{-1}t), \quad \Psi_2{}^* = \max_{0\leqslant t\leqslant T}\Psi_2(t)$$

Corollary:

1) $\mathbf{v}^{(m)}$ (or $\mathbf{v}^{\cdot(m)}$) remains bounded in $L_\infty(0, T_m; V_0)$ (or in $L_\infty(0, T_m; H_0)$).

2) The solution $\mathbf{v}^{(m)}$ can be continued to some time segment T for any m.

3) We have for the functions $\mathbf{v}^{(m)}$ and $\mathbf{v}^{\cdot(m)}$

$$\mathbf{v}^{(m)} \in L_\infty(0, T; V_0), \quad \mathbf{v}^{\cdot(m)} \in L_\infty(0, T; H_0) \tag{2.15}$$

Convergence of a sequence of approximate solutions. It follows from (2.15) that a subsequence $\mathbf{v}^{(\mu)}$ can be extracted from the sequence $\mathbf{v}^{(m)}$ such that $\mathbf{v}^{(\mu)}$ $(\mathbf{v}^{\cdot(\mu)})$ will converge weakly to $\mathbf{v}$ $(\mathbf{v}^\cdot)$ in $L_\infty(0,T; V_0)$ (or in $L_\infty(0, T; H_0)$).

Let us show that $\mathbf{v}(t)$ satisfies the equation and intitial conditions (1.7). We introduce the space of functions

$$G = \left\{\varphi : \varphi = \sum_{i=1}^{\mu_0} \varphi_i(t)\mathbf{w}_i, \ \varphi_i \in C^1([0, T]), \varphi_i(T) = 0\right\}$$

Here μ_0 is a finite integer. Replacing $\delta\mathbf{v}$ by φ in (2.6) and integrating the expression obtained, we find

$$\int_0^T \{-(\mathbf{v}^{\cdot(\mu)}, \varphi^\cdot) + (\nabla \mathrm{E}[\mathbf{v}^{(\mu)} + \mathbf{u}_0], \varphi) - (\mathbf{f}_0, \varphi) - (\mathbf{F}, \varphi)_\Gamma\}\, dt = (\mathbf{v}_0^{\cdot(\mu)}, \varphi(0)), \quad \forall\varphi \in G \quad (\mu = m \geqslant \mu_0) \tag{2.16}$$

Passing to the limit in μ in (2.16), we obtain

$$\int_0^T \{-(\mathbf{v}^\cdot, \varphi^\cdot) + (\nabla \mathrm{E}[\mathbf{v} + \mathbf{u}_0], \varphi) - (\mathbf{f}_0, \varphi) - (\mathbf{F}, \varphi)_\Gamma\}\, dt = (\mathbf{v}_0^\cdot, \varphi(0)), \quad \forall\varphi \in G \tag{2.17}$$

Since finite linear combinations of $\mathbf{w}_i$ are compact in V_0, then (2.17) is valid for any $\varphi \in C^1([0, T]; V_0)$, $\varphi(T) = 0$. Therefore, the equality

$$\mathbf{v}^{\cdot\cdot} + \nabla \mathrm{E}[\mathbf{v} + \mathbf{u}_0] = \Phi \quad ((\Phi, \varphi) \equiv (\mathbf{f}_0, \varphi) + (\mathbf{F}, \varphi)_\Gamma) \tag{2.18}$$

is valid in the sense of distributions in $[0, T]$ with values in V_0. It hence follows that $\mathbf{v}^{\cdot\cdot} \in L_\infty(0, T; V')$. We see that $\mathbf{v}(t)$ satisfies the initial conditions (1.7). Let us compare the scalar product of (2.18) and $\varphi \in G$ with (2.17). We obtain

$$(\mathbf{v}_0^\cdot, \varphi(0)) = (\mathbf{v}^\cdot(0), \varphi(0)) \quad \forall\varphi \in G$$

Therefore, $\mathbf{v}^\cdot(0) = \mathbf{v}_0^\cdot$. There results from the convergence of the sequence of approximate solutions that $\mathbf{v}^{(\mu)} = \mathbf{v}_0 \to \mathbf{v}(0)$, which means that $\mathbf{v}(0) = \mathbf{v}_0$. Since $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$, we arrive at the deduction that $\mathbf{u}$ satisfies all the conditions of the theorem.

3. Uniqueness of the solutions. Let us prove two theorems

setting forth the uniqueness of the solutions in the case of stationarity of the solutions and in the case of sufficient smoothness of the solutions, namely, $\mathbf{u}$, $\mathbf{u}^{\cdot} \in H_1(\Omega)$.

We introduce the Hamilton functional

$$H = {}^1/_2(\mathbf{p}, \mathbf{p}) + \mathrm{E}[\mathbf{v} + \mathbf{u}_0] - (\boldsymbol{\Phi}(t), \mathbf{v}) \tag{3.1}$$

Here $\mathbf{p} = \mathbf{v}^{\cdot}$, $(\boldsymbol{\Phi}, \mathbf{v}) = (\mathbf{f}_0, \mathbf{v}) + (\mathbf{F}, \mathbf{v})_\Gamma$, $\forall \mathbf{v} \in V_0$. Let us replace the variational equation (1.7) by the canonical equations

$$\begin{aligned} d\mathbf{p}/dt &= -\nabla_v H \equiv -\nabla \mathrm{E}[\mathbf{v} + \mathbf{u}_0] + \boldsymbol{\Phi}(t) \\ d\mathbf{v}/dt &= \nabla_p H \equiv \mathbf{p} \end{aligned} \tag{3.2}$$

The first equation in (3.2) is considered in the space $H_{-1}(\Omega)$ and the second in $H_0(\Omega)$.

We assume that $(\mathbf{p}, \mathbf{v})$, $(\mathbf{p} + \Delta\mathbf{p}, \mathbf{v} + \Delta\mathbf{v})$ are two solutions of (3.2) corresponding to the same initial conditions ($\Delta\mathbf{p}$ and $\Delta\mathbf{v}$ are zero at the initial instant). Then the functions $\Delta\mathbf{p}$ and $\Delta\mathbf{v}$ are a solution of the system of equations

$$\begin{aligned} d\Delta\mathbf{p}/dt &= -(\nabla \mathrm{E}[\mathbf{u} + \Delta\mathbf{v}] - \nabla \mathrm{E}[\mathbf{u}]) \\ d\Delta\mathbf{v}/dt &= \Delta\mathbf{p} \quad (\mathbf{u} = \mathbf{v} + \mathbf{u}_0) \end{aligned} \tag{3.3}$$

Let us define the functional $W(\Delta\mathbf{p}, \Delta\mathbf{v}, t)$ according to the equality

$$W = {}^1/_2(\Delta\mathbf{p}, \Delta\mathbf{p}) + \mathrm{E}[\mathbf{u} + \Delta\mathbf{v}] - \mathrm{E}[\mathbf{u}] - (\nabla \mathrm{E}[\mathbf{u}], \Delta\mathbf{v}) \tag{3.4}$$

We note that it is a Hamilton functional for (3.3), which means that its total derivative has the following form because of (3.3)

$$dW/dt = \partial W/\partial t = (\nabla \mathrm{E}[\mathbf{u} + \Delta\mathbf{v}] - \nabla \mathrm{E}[\mathbf{u}], \mathbf{u}^{\cdot}) - (\nabla^2 \mathrm{E}[\mathbf{u}]\mathbf{u}^{\cdot}, \Delta\mathbf{v}) \tag{3.5}$$

T h e o r e m (s t a t i o n a r y c a s e). Let the solution $\mathbf{v}$ of (1.7) be such that $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$ is independent of the time, and the functional $\mathrm{E}[\mathbf{u}]$ is convex, i.e.,

$$\begin{aligned} &\mathrm{E}[\mathbf{u} + \Delta\mathbf{v}] - \mathrm{E}[\mathbf{u}] - (\nabla \mathrm{E}[\mathbf{u}], \Delta\mathbf{v}) \geqslant \alpha \|\Delta\mathbf{v}\|_1^2 \\ &(\alpha > 0, \|\Delta\mathbf{v}\|_1 < h, h > 0) \end{aligned} \tag{3.6}$$

Then the solution $\mathbf{v}$ is unique.

N o t e 4. The stationary solution corresponds either to the equilibrium position of the system for which the boundary conditions (1.5) and the external forces in (1.4) must be assumed stationary, or to the case when the deformed body is rigidly displaced.

N o t e 5. Condition (3.6) can be replaced by a condition on the second Frechet variation of the functional $\mathrm{E}[\mathbf{u}]$:

$$\|\mathbf{w} - \mathbf{u}\|_1 < h\,(h > 0), \quad (\nabla^2 \mathrm{E}[\mathbf{w}]\Delta\mathbf{v}, \Delta\mathbf{v}) \geqslant 2\alpha \|\Delta\mathbf{v}\|_1^2$$

In fact, let $F_1(\tau)$: $[0, 1] \to R^1$ and $F_1(\tau) = \mathrm{E}[\mathbf{u} + \tau\Delta\mathbf{v}] - \mathrm{E}[\mathbf{u}] - (\nabla \mathrm{E}[\mathbf{u}], \tau\Delta\mathbf{v})$. We have

$$\begin{aligned} &dF_1(\tau)/d\tau = (\nabla \mathrm{E}[\mathbf{u} + \tau\Delta\mathbf{v}] - \nabla \mathrm{E}[\mathbf{u}], \Delta\mathbf{v}) \\ &\|\Delta\mathbf{v}\|_1 < h, \ d^2F_1(\tau)/d\tau^2 = (\nabla^2 \mathrm{E}[\mathbf{u} + \tau\Delta\mathbf{v}]\Delta\mathbf{v}, \Delta\mathbf{v}) \geqslant 2\alpha \|\Delta\mathbf{v}\|_1^2 \end{aligned} \tag{3.7}$$

Integrating the second equation in (3.7) twice with respect to τ, we obtain (3.6).

P r o o f o f t h e t h e o r e m. Since $\mathbf{u}$ is independent of the time, the functional W defined by (3. 4), also does not contain the time explicitly. It then follows from (3. 5) that the functional $W(\Delta\mathbf{p}, \Delta\mathbf{v})$ is constant and equal to zero since $\Delta\mathbf{p} = \Delta\mathbf{v} = 0$ at the initial time. Because of (3. 6), we have for $\|\Delta\mathbf{v}\|_1 < h$

$$W(\Delta\mathbf{p}, \Delta\mathbf{v}) \geqslant {}^1/_2 \|\Delta\mathbf{p}\|_0^2 + \alpha\|\Delta\mathbf{v}\|_1^2$$

Therefore for $t \in [0, T]$ we have $\Delta\mathbf{p} = \Delta\mathbf{v} = 0$ and the solution is unique.

T h e o r e m (d y n a m i c c a s e). If

$$\mathbf{u}^\cdot = \mathbf{v}^\cdot + \mathbf{u}_0^\cdot \in H_1(\Omega), \quad \|\mathbf{u}^\cdot\|_1 < c_2 \tag{3.8}$$
$$\mathbf{w} \in V_0, \quad \|\mathbf{w}\|_1 < c_3, \quad (\nabla^3\mathrm{E}[\mathbf{w}](\Delta\mathbf{v}, \Delta\mathbf{v}), \mathbf{u}^\cdot) \leqslant M\|\Delta\mathbf{v}\|_1^2$$

where c_2, c_3, M are positive constants and condition (3. 6) is valid, then the solution $\mathbf{v}(t)$ of (1. 7) is unique.

N o t e 6. The first condition in (3. 8) is satisfied if $\mathbf{v}^\cdot \in H_1(\Omega)$ since $\mathbf{u}_0^\cdot \in V_0$ according to the Note 2.

P r o o f o f t h e t h e o r e m. Let us consider the function

$$F_2(\tau) = (\nabla\mathrm{E}[\mathbf{u} + \tau\Delta\mathbf{v}] - \nabla\mathrm{E}[\mathbf{u}], \mathbf{u}^\cdot) -$$
$$(\nabla^2\mathrm{E}[\mathbf{u}]\,\mathbf{u}^\cdot, \tau\Delta\mathbf{v}) \quad (0 \leqslant \tau \leqslant 1)$$

for which the following relations are valid:

$$F_2(0) = 0, \; F_2(1) = \partial W / \partial t, \quad dF_2(0) / d\tau = 0 \tag{3.9}$$
$$d^2F_2 / d\tau^2 = (\nabla^3\mathrm{E}[\mathbf{u} + \tau\Delta\mathbf{v}](\Delta\mathbf{v}, \nabla\mathbf{v}), \mathbf{u}^\cdot)$$

Here $\nabla^2\mathrm{E}[\mathbf{u}]$ and $\nabla^3\mathrm{E}[\mathbf{u}]$ are the second and third Frechet differentials of the functional $\mathrm{E}[\mathbf{u}]$ [6].

Estimating the right side of the last relationship in (3. 9) with (3. 8) taken into account, we arrive at the inequality

$$d^2F_2 / d\tau^2 \leqslant M\|\Delta\mathbf{v}\|_1^2$$

Integrating twice with respect to τ, we obtain

$$F_2(1) = \partial W / \partial t \leqslant {}^1/_2 M\|\Delta\mathbf{v}\|_1^2$$

Integrating of (3. 5) yields

$$W(t) - W(0) \leqslant \frac{1}{2} M \int_0^t \|\Delta\mathbf{v}\|_1^2 dt \tag{3.10}$$

We note that $W(0) = 0$, and according to (3. 6) we strengthen the inequality (3. 10):

$$\frac{1}{2}\|\Delta\mathbf{p}\|_0^2 + \alpha\|\Delta\mathbf{v}\|_1^2 \leqslant \frac{M}{2\alpha}\int_0^t \left(\frac{1}{2}\|\Delta\mathbf{p}\|_0^2 + \alpha\|\Delta\mathbf{v}\|_1^2\right) dt$$

There follows from the Gronwall inequality

$$t \in [0, T], \quad {}^1/_2\|\Delta\mathbf{p}\|_0^2 + \alpha\|\Delta\mathbf{v}\|_1^2 \leqslant 0$$

This latter is possible for $\Delta\mathbf{p} = \Delta\mathbf{v} = 0$, therefore, the solution is unique.

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